Round of Choices

October 24th, 2020

Set 1

1. Let S be a set of 2020 distinct real numbers. For each real number $r \in S$, let its *best friend* be the number in S closest to r that is not r, assuming there are no ties. A number is *popular* if it is the best friend of at least two other numbers. If the number of popular numbers cannot exceed N, find the smallest possible value of N.

2. We construct the *Determinant-1 Triangle* as follows:

   - In the nth row, the first and last entries equal 1.
   - Let d an entry of the nth row other than a first or last entry. Let b and c be the two entries in the previous row directly above d and let a be the entry that is directly above both b and c. Then $ad - bc = 1$.

   The first 5 rows of the Determinant-1 Triangle are shown below.

Row 1:					1					
Row 2:				1		1				
Row 3:			1		2		1			
Row 4:		1		3		3		1		
Row 5:	1		4		5		4		1	

   Note that $5 \cdot 2 - 3 \cdot 3 = 1$. Given that all entries are integral, of the first 51 rows, how many entries are 1 more than a power of 2 (1 is a considered a power of 2)?

3. Let Γ_1 and Γ_2 be two circles with radii 37 and 20, and centers O_1 and O_2, respectively. Suppose they intersect at points R and S. Denote by ω the circle with diameter RS. Suppose O_1S and O_2S meet ω at P and Q, respectively. Given that $O_1O_2 = 51$, the square root of the area of triangle SPQ can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b and c such that no square divides b. Compute $a + b + c$.

Set 2

1. Banjemin starts with two positive integers on a blackboard. Every minute, he changes the two numbers a and b to $a + \gcd(a, b)$ and $b + \gcd(a, b)$. If Banjemin started with the numbers 1017 and 2020, what is the greatest common divisor of the two numbers on the blackboard after a year?

2. The following sum
$$\sum_{i=0}^{\infty} \left\lfloor \frac{i}{3} \right\rfloor \frac{1}{4^i}$$
may be expressed in the form m/n where m and n are relatively prime positive integers. Compute $m + n$.

3. Billy draws the points (x, x^3) for $x \in \{-10, -9, \ldots, 9, 10\}$. How many distinct lines can Jimmy draw that pass through at least two of Billy's points?

Set 3

1. Gamblers Kodvick, Maverick, Rodrick and John Wick decide to play a game. They each have a debt of d_1, d_2, d_3, d_4 in that order, respectively, with $0 < d_1 < d_2 < d_3 < d_4$. In each round of the game, the gamblers are randomly ranked with every possible ranking being equally likely. At the end of the round, the winner swaps debts with the person with least debt and the loser swaps debts with the person with most debt. However, if the winner of the round already has the least debt then the game is over and the round winner is declared the winner of the game. The probability that Kodvick wins the game may be expressed as m/n for relatively prime positive integers m and n. Compute $m + n$.

2. Find the sum of all possible values of $|x - y|$ over all ordered pairs (x, y) with $1 \le x, y \le 50$ such that $x^2 + y > 3$ is prime and $x^2 + y$ divides $x^3 + y^3$.

3. Let ABC be a triangle with $AB = 12$, $BC = 14$ and $CA = 16$. Let D, E, and F be points on sides BC, CA and AB, respectively such that $AF = BD = CE = 4$. The lines ℓ_D, ℓ_E, and ℓ_F passing through D, E, and F, perpendicular to sides BC, CA and AB, respectively, determine a triangle XYZ. The area of triangle XYZ may be expressed as $a\sqrt{b}/c$ for relatively prime integers a and c such that b is not divisible by the square of any prime. What is $a + b + c$?

Set 4

1. How many points on the Cartesian plane (x, y) satisfy $((x-20)^{20}-20)^{20}+((y-20)^{20}-20)^{20}=0$?

2. Monkeys have created their own alphabet which consists of 3 letters: A, B, and N. Two words have the same meaning if one of them can be constructed from the other by replacing any AA with A, replacing any BB with B, replacing any NN with N, replacing any AB with BA, replacing any BN with NB, or replacing any NA with AN. For example $BANANA$ has the same meaning as $BNAANA$ which also has the same meaning as $BNANA$. How many different meanings can the monkeys express with non-empty words?

3. Consider all possible values of x for which x is a real number satisfying

$$2\max\left(\frac{x}{5}, \sqrt{|x|}\right)^2 - 3\min\left(\frac{x}{5}, \sqrt{|x|}\right) = 1,$$

where $\max(a, b)$ denotes the larger of the two numbers a and b, and $\min(a, b)$ denotes the smaller. If the sum of these values can be expressed as m/n where m and $n > 0$ are relatively prime integers, find $m + n$.

Set 5

1. Alex is fond of the word "abc", he likes it so much that he counts the number of subsequences of "abc" in any given word. For example, when Alex is given the word "ababc", he counts 3. Given that he is given a random word of length 10, consisting of the characters 'a','b', and 'c', if the expected number of subsequences of "abc" he counts can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

2. Banjemin possesses a *Kodvix Square*. Below are four different states of the Kodvix square, with the first one being the "solved" state.

1	4	7
2	5	8
3	6	9

3	4	1
2	5	8
9	6	7

1	5	7
8	6	2
3	4	9

1	6	7
5	8	2
3	4	9

There are three fundamental moves.

- The *Cyclic Shift*, which rotates the corners of the square clockwise. In other words, it is the sequence of swaps of two grid elements required to get from the first state to the second.
- The *R-shift*, which is the sequence of swaps of two grid elements required to get from the first state to the third.
- The *S-shift*, which is the sequence of swaps of two grid elements required to get from the first state to the fourth.

Banjemin may scramble the Kodvix square using as many fundamental moves as he wishes (swapping two grid elements is NOT a fundamental move). How many distinct scrambled squares can he end up with? (States that can be rotated to match the other are considered distinct)

3. Suppose for two integers x and y, $3x^2 + 3x + 1641 = 9y^3$. Find the product of all possible values of $|x|$.

Set 6

1. Compute the number of pairs of integers (x, y) with $1 \leq x < y \leq 120$ such that 2^{x^2} and 2^{y^2} leave the same remainder upon division by 455.

2. In triangle ABC, the incircle touches AC and AB at E and F, respectively. Let M be the midpoint of BC. Suppose $BFEM$ is a cyclic quadrilateral and $AC - AB = 38$. Then, the length of BC can be expressed as $a + b\sqrt{c}$ where a, b, c are positive integers such that c is not divisible by the square of any prime. What is $a + b + c$?

3. Alex is fond of the word "abc". He likes it so much that he is happy if there is at least one subsequence of "abc" in any given word. How many distinct strings of length 8 consisting of characters 'a','b', and 'c' will make Alex happy?

Set 7

1. In $\triangle ABC$, $AB = 13$, $AC = 15$, and $BC = 14$. Let the perpendicular bisector of BC meet line AB at D and line AC at E. Let the circumcircle of $\triangle ADE$ meet the circumcircle of $\triangle ABC$ at X. If $\frac{DX}{EX}$ is expressed as m/n where m and n are relatively prime postive integers. What is $m + n$?.

2. Call a pair of integers (i, j) perfect if $13 \leq i < j \leq 169$ and $\binom{i}{13} + \binom{i+1}{13} + \binom{i+2}{13} + \ldots + \binom{j-1}{13}$ is divisible by 13. How many perfect pairs are there?

3. Given that
$$\sum_{n=3}^{\infty} \arctan\left(\frac{10}{25n^2 + 25n + 4}\right) = \arctan\left(\frac{m}{n}\right)$$
for relatively prime positive integers m and n, compute $m + n$.